New Determination of the Fine Structure Constant from the Electron g Value and QED

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Quantum electrodynamics (QED) predicts a relationship between the dimensionless magnetic moment of the electron (g) and the fine structure constant (α). A new measurement of g using a one-electron quantum cyclotron, together with a QED calculation involving 891 eighth-order Feynman diagrams, determine $\alpha^{-1} = 137.035\,999\,710(96)$ [0.70 ppb]. The uncertainties are 10 times smaller than those of nearest rival methods that include atom-recoil measurements. Comparisons of measured and calculated g test QED most stringently, and set a limit on internal electron structure.

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The electron g value [1], the dimensionless measure of the electron magnetic moment in terms of the Bohr magneton, is a fundamental property of the simplest of stable elementary particles. The fine structure constant,

$$\alpha = \frac{1}{4\pi\epsilon_0}\frac{e^2}{\hbar c}, \qquad (1)$$

gives the strength of the electromagnetic interaction, and is a crucial building block in our system of fundamental constants [2]. Quantum electrodynamics (QED), the wonderfully successful theory that describes the interaction of light and matter, provides an extremely precise prediction for the relationship between g and α, with only small, well-understood corrections for short-distance physics.

A new measurement of g [1] achieves an uncertainty that is nearly 6 times smaller than that of the last measurement of g, reported back in 1987 [3]. An improved QED calculation that includes contributions from 891 eighth-order Feynman diagrams [4] now predicts g in terms of α through order $(\alpha/\pi)^4$. Together, the measured g and the QED calculation make it possible for experimenters and theorists to jointly announce here a new determination of α. Its 0.70 ppb uncertainty is the first reduction in uncertainty for an α determination since 1987 (Fig. 1). The uncertainty is 10 times smaller than for any other method to determine α [5,6]. Comparisons of measured and calculated g values test QED most stringently, and probe for possible electron substructure at a surprisingly high energy scale.

Since $g = 2$ for a Dirac point particle, the dimensionless moment is often written as $g = 2(1 + a)$. The deviation a is called the anomalous magnetic moment of the electron or simply the electron anomaly. It arises from the vacuum fluctuations and polarizations that are described by QED, with only small additions for short-distance physics that are well understood within the standard model [7],

$$a = a(\mathrm{QED}) + a(\mathrm{hadron}) + a(\mathrm{weak}). \qquad (2)$$

Any additional contribution to the anomaly would therefore be extremely significant, indicating electron substructure [8], new short-distance physics, or problems with QED theory (and perhaps with quantum field theory more generally).

A long series of improved measurements of g, reviewed in [3,9], now continues after a hiatus of nearly 20 years. The new measurement achieves a much smaller uncertainty in g [1] by resolving the quantum cyclotron and spin levels of one electron [10] suspended for months at a time in a cylindrical Penning trap [11]. Quantum jump spectroscopy of transitions between these levels determines the spin and cyclotron frequencies, and $g/2$ is essentially the ratio of such measured frequencies. The cylindrical Penning cavity [11] shapes the radiation field in which the electron is located, narrowing resonance linewidths by inhibiting spontaneous emission, and providing boundary conditions which make it possible to identify the symmetries of cavity radiation modes [12]. A quantum

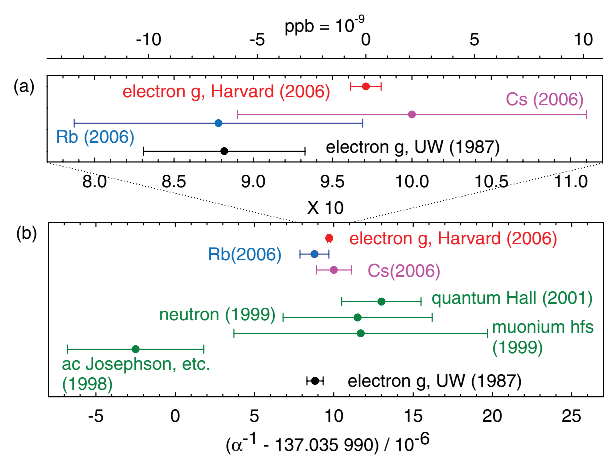

FIG. 1 (color). The least uncertain α determinations [3,5,6] (a), with older determinations [2] on a 10 times larger scale (b). Measured g are converted to α using current QED theory.

nondemolition coupling of the cyclotron and spin energies to the frequency of an orthogonal and nearly harmonic electron oscillation, reveals the quantum state [10]. This harmonic oscillation of the electron is self-excited [13], by a signal derived from its own motion, to produce the large signal-to-noise ratio needed to quickly read out the quantum state without ambiguity.

The preceding Letter [1] reports the new measurement,

$$g/2 = 1.001\,159\,652\,180\,85\,(76)\ [0.76\ \text{ppt}]. \tag{3}$$

The largest contribution to the 7.6×10^{-13} uncertainty arises from an imperfect line shape model (0.6 ppt); likely this can be understood and reduced with careful study. Extremely small magnetic field instabilities are one possible cause. The second source of uncertainty is cavity shifts (0.4 ppt), caused when the cyclotron frequency of an electron in the trap cavity is shifted by interactions with cavity radiation modes that are near in frequency. The frequencies of cavity radiation modes are measured well enough to identify the spatial symmetry of the modes, and to calculate and correct for cavity shifts to g from the known electromagnetic field configurations. A smaller third uncertainty is statistical (0.2 ppt), and could be reduced with more measurements.

QED calculations involving many Feynman diagrams provide the coefficients for expansions in powers of the small ratio $\alpha/\pi \approx 2 \times 10^{-3}$. The QED anomaly

$$a(\text{QED}) = A_1 + A_2(m_e/m_\mu) + A_2(m_e/m_\tau) + A_3(m_e/m_\mu, m_e/m_\tau), \tag{4}$$

is a function of lepton mass ratios. Each A_i is a series,

$$A_i = A_i^{(2)}\left(\frac{\alpha}{\pi}\right) + A_i^{(4)}\left(\frac{\alpha}{\pi}\right)^2 + A_i^{(6)}\left(\frac{\alpha}{\pi}\right)^3 + \ldots. \tag{5}$$

The calculations are so elaborate that isolating and eliminating mistakes is a substantial challenge, as is determining and propagating numerical integration uncertainties.

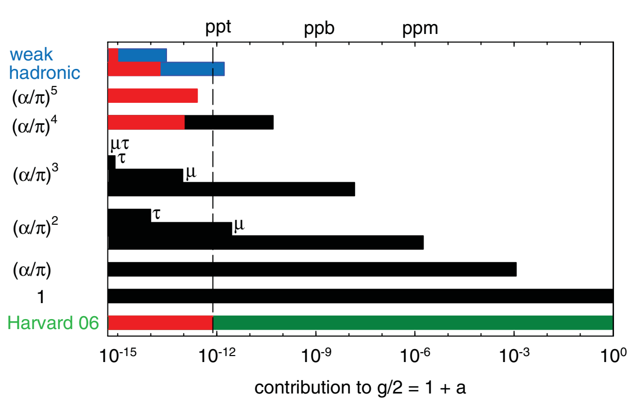

FIG. 2 (color). Contributions to $g/2$ for the experiment (green), terms in the QED series (black), and from short-distance physics (blue). Uncertainties are in red. The μ, τ, and $\mu\tau$ indicate terms dependent on mass ratios m_e/m_μ, m_e/m_τ and the two ratios, m_e/m_μ and m_e/m_τ, respectively.

Figure 2 compares the contributions and uncertainties for $g/2$. The leading constants for second [14], fourth [15–17], and sixth [18–22] orders,

$$A_1^{(2)} = 0.5, \tag{6}$$

$$A_1^{(4)} = -0.328\,478\,965\,579\ldots, \tag{7}$$

$$A_1^{(6)} = 1.181\,241\,456\,587\ldots, \tag{8}$$

have been evaluated exactly. The latter confirms the value 1.181 259(4) obtained numerically [23]. Very small mass-dependent QED additions [24–29],

$$\begin{aligned} A_2^{(4)}(m_e/m_\mu) &= 5.197\,386\,70\,(28) \times 10^{-7}, \\ A_2^{(4)}(m_e/m_\tau) &= 1.837\,62\,(60) \times 10^{-9}, \\ A_2^{(6)}(m_e/m_\mu) &= -7.373\,941\,64\,(29) \times 10^{-6}, \\ A_2^{(6)}(m_e/m_\tau) &= -6.581\,9\,(19) \times 10^{-8}, \\ A_3^{(6)}(m_e/m_\mu, m_e/m_\tau) &= 0.190\,945\,(62) \times 10^{-12}, \end{aligned} \tag{9}$$

are exactly known functions of the lepton mass ratios [30], from which they derive their uncertainty

Crucial progress came in evaluating, checking, and determining the uncertainty in the eighth order $A_1^{(8)}$, which includes contributions of 891 Feynman diagrams. Typical diagrams of the 13 gauge invariant subgroups are shown in Fig. 3. Integrals of 373 of these have been verified (and corrected) by more than one independent formulation [4,31]. Verification of the 518 diagrams with no closed lepton loops is in progress using an automating algorithm [32]. Their renormalization terms are derived by systematic reduction of original integrands applying a simple power-counting rule [33], allowing extensive cross-checking among themselves and with exactly known diagrams of lower order [34]. Numerical integrations with VEGAS [35], on many supercomputers over more than 10 years, then yields [4]

$$A_1^{(8)} = -1.7283\,(35). \tag{10}$$

The uncertainty, determined using estimated errors from

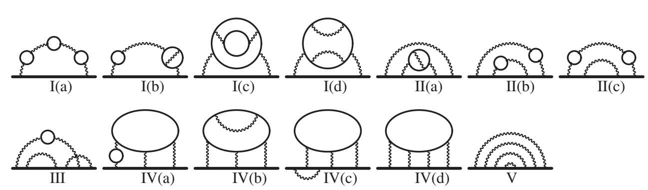

FIG. 3. Typical diagrams from each gauge invariant subgroup that contributes to the eighth-order electron magnetic moment. Solid and wiggly curves represent the electron and photon, respectively. Solid horizontal lines represent the electron in an external magnetic field.

VEGAS, is improved by an order of magnitude over the previous value [36].

The high experimental precision makes the tenth-order contribution to g potentially important if the unknown $A_1^{(10)}$ is unexpectedly large, though this seems unlikely. To get a feeling for its possible impact we use a bound

$$|A_1^{(10)}| < x \tag{11}$$

with an estimate $x = 3.8$ [2], while awaiting a daunting evaluation of contributions from 12 672 Feynman diagrams that is now underway [4,32].

Also owing to the high-precision, non-QED contributions,

$$\begin{aligned} a(\text{hadron}) &= 1.671\,(19) \times 10^{-12}, \\ a(\text{weak}) &= 0.030\,(01) \times 10^{-12}, \end{aligned} \tag{12}$$

must be included. Fortunately, these are small and well understood in the context of the standard model [2,7].

The newly measured g [1], and the high-precision QED calculation, together determine a value of α that has an uncertainty 10 times smaller than any other method to determine α,

$$\alpha^{-1}(\text{H06}) = 137.035\,999\,710\,(12)\,(30)\,(90) \tag{13}$$

$$= 137.035\,999\,710\,(96)\ [0.70\ \text{ppb}]. \tag{14}$$

In the first line, the first uncertainty is from the calculated $A_1^{(8)}$ and the last is from the measured g. The middle uncertainty is from the estimated bound on the unknown $A_1^{(10)}$ in Eq. (11). It would be 8 for a bound $x = 1$. It is $8x$ more generally, rounding to 30 for the estimate $x = 3.8$ [2]. Without the exact $A_1^{(6)}$ of Laporta and Remiddi, in Eq. (8) and [22], numerical inaccuracy [23] would add a fourth uncertainty (60) to the list of three in Eq. (13).

The best determinations of α are compared in Fig. 1. The least uncertain values independent of the electron g [5,6]

$$\alpha^{-1}(\text{Cs06}) = 137.036\,000\,00\,(110)\ [8.0\ \text{ppb}], \tag{15}$$

$$\alpha^{-1}(\text{Rb06}) = 137.035\,998\,78\,(91)\ [6.7\ \text{ppb}], \tag{16}$$

rely upon many experiments, including the measured Rydberg constant [37], the Cs or Rb mass in amu [38], and the electron mass in amu [39,40]. The needed $\hbar/M[\text{Cs}]$ comes from an optical measurement of the Cs D1 line [6,41], and the "preliminary" recoil shift for a Cs atom in an atom interferometer [42]. The needed $\hbar/M[\text{Rb}]$ come from a measurement of an atom recoil of a Rb atom in an optical lattice [5].

The most stringent test of QED comes from comparing the α from g and QED, with the independent values

$$\alpha^{-1}(\text{Cs}) - \alpha^{-1}(\text{H06}) = 0.29\,(1.10) \times 10^{-6}, \tag{17}$$

$$\alpha^{-1}(\text{Rb}) - \alpha^{-1}(\text{H06}) = -0.93\,(0.92) \times 10^{-6}. \tag{18}$$

Good agreement, within 0.3 and 1.0 standard deviations, respectively, gives no indication of a QED breakdown. Equivalent comparisons of measured and "calculated" magnetic moments (the latter using a measured α as an input),

$$a(\text{Cs06}) - a(\text{H06}) = -2.5\,(9.3) \times 10^{-12}, \tag{19}$$

$$a(\text{Rb06}) - a(\text{H06}) = 7.9\,(7.7) \times 10^{-12}, \tag{20}$$

are traditionally used for the QED test, and for limits on electron substructure [8]. The uncertainties in the comparisons come entirely from $\alpha[\text{Cs}]$ and $\alpha[\text{Rb}]$, better measurements of which are badly needed. The much smaller uncertainties in the measured g and QED would allow a 10 times more stringent test of QED.

Comparing experiment and theory probes for possible electron substructure at an energy scale one might only expect from a large accelerator. An electron whose constituents would have mass $m^* \gg m$ has a natural size scale, $R = \hbar/(m^* c)$. The simplest analysis of the resulting magnetic moment [8] gives $\delta a \sim m/m^*$, suggesting that $m^* >$ 34 000 TeV/c^2 and $R < 6 \times 10^{-24}$ m. This would be an incredible limit, since the largest e^+e^- collider (LEP) probes for a contact interaction at an $E = 10.3$ TeV [43], with $R < (\hbar c)/E = 2 \times 10^{-20}$ m.

However, the simplest argument also implies that the first-order contribution to the electron self-energy goes as m^* [8]. Without heroic fine tuning (e.g., the bare mass canceling this contribution to produce the small electron mass) some internal symmetry of the electron model must suppress both mass and moment. For example, a chirally invariant model [8], leads to $\delta a \sim (m/m^*)^2$. In this case, $m^* > 130$ GeV/c^2 and $R < 1 \times 10^{-18}$ m. These limits seem remarkable for an experiment carried out at 100 mK, although they do not compete with LEP. If this test was limited only by the experimental uncertainty in a, then we could set a limit $m^* > 600$ GeV.

What theory improvements might be expected in the future? The theory contribution to the uncertainty in the new α is already less than that from experiment by a factor of 3. The eighth-order uncertainty in $A_1^{(8)}$ can be reduced with the accumulation of better statistics in the numerical evaluation of integrals. Ambitious efforts underway aim for a complete analytic evaluation, thereby entirely removing this uncertainty [44]. A calculation of the tenth-order coefficient, $A_1^{(10)}$, is needed if an α with smaller uncertainties is ever to be deduced from a better g. The evaluation is a formidable challenge given the mentioned contributions from 12 672 Feynman diagrams. Considerable progress in setting up and integrating many of these diagrams has been reported [32,45]. It now seems feasible to evaluate $A_1^{(10)}$ to a few percent.

What experimental improvements can be expected? Experiments underway aim to substantially reduce the uncertainty in atom-recoil measurements that currently contribute the largest uncertainty to independent determinations of α [46,47]. The preceding Letter [1] mentions new methods that may further reduce the uncertainty in the electron g. This fully quantum measurement has only been recently realized, so much remains to be explored and optimized.

In conclusion, the fine structure constant is determined with much smaller uncertainty than in the past by a new measurement of the electron g and improved QED theory. The absence of electron substructure is assumed, and only small corrections for short-distance scale physics are needed. The new α has a 10 times smaller uncertainty than that from any other method. Comparing the α from g and QED, to the α determined independently with Cs and Rb, shows that QED continues to be a superb description of the interaction of light and matter. A QED test that is 10 times more stringent is possible with the current uncertainties in g and the QED calculation, if ever an α independent of g is determined with the uncertainty reported here. Comparing the measured and calculated g sets a limit on possible electric substructure at the 130 GeV level, again limited by the uncertainty in independent determinations of α, not by uncertainties in g or QED calculations.

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[1] B. Odom, D. Hanneke, B. D'Urso, and G. Gabrielse, Phys. Rev. Lett. **97**, 030801 (2006).
[2] P. J. Mohr and B. N. Taylor, Rev. Mod. Phys. **77**, 1 (2005).
[3] R. S. Van Dyck, Jr., P. B. Schwinberg, and H. G. Dehmelt, Phys. Rev. Lett. **59**, 26 (1987).
[4] T. Kinoshita and M. Nio, Phys. Rev. D **73**, 013003 (2006).
[5] P. Cladé, E. de Mirandes, M. Cadoret, S. Guellati-Khélifa, C. Schwob, F. Nez, L. Julien, and F. Biraben, Phys. Rev. Lett. **96**, 033001 (2006).
[6] V. Gerginov, K. Calkins, C. E. Tanner, J. McFerran, S. Diddams, A. Bartels, and L. Hollberg, Phys. Rev. A **73**, 032504 (2006).
[7] A. Czarnecki, B. Krause, and W. J. Marciano, Phys. Rev. Lett. **76**, 3267 (1996).
[8] S. J. Brodsky and S. D. Drell, Phys. Rev. D **22**, 2236 (1980).
[9] A. Rich and J. C. Wesley, Rev. Mod. Phys. **44**, 250 (1972).
[10] S. Peil and G. Gabrielse, Phys. Rev. Lett. **83**, 1287 (1999).
[11] G. Gabrielse and F. C. MacKintosh, Int. J. Mass Spectrom. Ion Processes **57**, 1 (1984).
[12] J. Tan and G. Gabrielse, Phys. Rev. Lett. **67**, 3090 (1991).
[13] B. D'Urso, R. Van Handel, B. Odom, D. Hanneke, and G. Gabrielse, Phys. Rev. Lett. **94**, 113002 (2005).
[14] J. Schwinger, Phys. Rev. **73**, 416L (1948).
[15] C. M. Sommerfield, Phys. Rev. **107**, 328 (1957).
[16] C. M. Sommerfield, Ann. Phys. (N.Y.) **5**, 26 (1958).
[17] A. Petermann, Helv. Phys. Acta **30**, 407 (1957).
[18] S. Laporta and E. Remiddi, Phys. Lett. B **265**, 182 (1991).
[19] S. Laporta, Phys. Rev. D **47**, 4793 (1993).
[20] S. Laporta and E. Remiddi, Phys. Lett. B **356**, 390 (1995).
[21] S. Laporta, Phys. Lett. B **343**, 421 (1995).
[22] S. Laporta and E. Remiddi, Phys. Lett. B **379**, 283 (1996).
[23] T. Kinoshita, Phys. Rev. Lett. **75**, 4728 (1995).
[24] M. A. Samuel and G. Li, Phys. Rev. D **44**, 3935 (1991).
[25] G. Li, R. Mendel, and M. A. Samuel, Phys. Rev. D **47**, 1723 (1993).
[26] A. Czarnecki and M. Skrzypek, Phys. Lett. B **449**, 354 (1999).
[27] S. Laporta, Nuovo Cimento A **106**, 675 (1993).
[28] S. Laporta and E. Remiddi, Phys. Lett. B **301**, 440 (1993).
[29] B. Lautrup, Phys. Lett. B **69**, 109 (1977).
[30] M. Passera, hep-ph/0606174.
[31] T. Kinoshita and M. Nio, Phys. Rev. Lett. **90**, 021803 (2003).
[32] T. Aoyama, M. Hayakawa, T. Kinoshita, and M. Nio, Nucl. Phys. **B740**, 138 (2006).
[33] P. Cvitanovic and T. Kinoshita, Phys. Rev. D **10**, 4007 (1974).
[34] T. Kinoshita, *Theory of the Anomalous Magnetic Moment of the Electron—Numerical Approach* (World Scientific, Singapore, 1990).
[35] G. P. Lepage, J. Comput. Phys. **27**, 192 (1978).
[36] V. W. Hughes and T. Kinoshita, Rev. Mod. Phys. **71**, S133 (1999).
[37] C. Schwob, L. Jozefowski, B. de Beauvoir, L. Hilico, F. Nez, L. Julien, F. Biraben, O. Acef, J. J. Zondy, and A. Clairon, Phys. Rev. Lett. **82**, 4960 (1999).
[38] M. P. Bradley, J. V. Porto, S. Rainville, J. K. Thompson, and D. E. Pritchard, Phys. Rev. Lett. **83**, 4510 (1999).
[39] T. Beier, H. Häffner, N. Hermanspahn, S. G. Karshenboim, H.-J. Kluge, W. Quint, S. Stahl, J. Verdú, and G. Werth, Phys. Rev. Lett. **88**, 011603 (2002).
[40] D. L. Farnham, R. S. Van Dyck, Jr., and P. B. Schwinberg, Phys. Rev. Lett. **75**, 3598 (1995).
[41] T. Udem, J. Reichert, R. Holzwarth, and T. W. Hänsch, Phys. Rev. Lett. **82**, 3568 (1999).
[42] A. Wicht, J. M. Hensley, E. Sarajlic, and S. Chu, Phys. Scr. **T102**, 82 (2002).
[43] D. Bourilkov, Phys. Rev. D **64**, 071701 (2001).
[44] S. Laporta and E. Remiddi (private communication).
[45] T. Kinoshita and M. Nio, Phys. Rev. D **73**, 053007 (2006).
[46] S. Chu (private communication).
[47] F. Biraben (private communication).